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Near-Best Approximate Solutions for a Class of Elliptic Partial Differential Equations

PETER A. MCCOY

Mathematics Department, United States Naval Academy, Annapolis, Maryland 21402, U.S.A.

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1. INTRODUCTION

A variety of steady state physical phenomena [3, 6–9] linked with boundary value problems for the diffusion equation, the wave equation in separated form, or those which arise after transforming the time dependence in an initial value problem are associated with linear second order partial differential equations of the form

$$\mathscr{L}(w) := [\nabla^2 + \mathbf{K} \cdot \mathbf{grad} + c(r, \theta)]\, w(r, \theta) = 0, \qquad (r, \theta) \in \Omega,$$

where

$$\nabla^2 := \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}, \tag{1}$$

and

$$\mathbf{K}(r, \theta) := \hat{e}_r G(r, \theta) + \hat{e}_\theta H(r, \theta), \qquad \mathbf{grad} := \hat{e}_r \partial_r + \hat{e}_\theta r^{-1}\partial_\theta.$$

Here, $\hat{e}_r$ and $\hat{e}_\theta$ are the standard basis vectors expressed in terms of the plane polar coordinates (r, θ). The coefficients are real analytic functions on the closure of the region Ω and $c(r, \theta) \leqslant 0$. In function theory, the solutions are viewed as natural extensions of harmonic or analytic functions, and Ω is taken as a bounded, star shaped region relative to the origin with a Hölder continuous boundary. Function theoretic methods [9, 10] that approximate solutions are of permanent interest and have been extensively developed for real-valued regular solutions of the Dirichlet problem

$$\text{(DP)} \quad \begin{array}{ll} \mathscr{L}(w) = 0, & (r, \theta) \in \Omega \\ w^-(r_0, \theta_0) = f(r_0 e^{i\theta_0}), & r_0 e^{i\theta_0} \in \partial\Omega \end{array} \tag{2}$$

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with continuous boundary data; $w^-(r_0, \theta_0) := \lim w(r, \theta)$ as $(r, \theta) \to (r_0, \theta_0)$ along the inner normal.

The focus of the function theoretic method is on a pair of integral operators that establish a one-to-one correspondence between solutions of the partial differential equation and associated functions that are analytic in a single complex variable. The point is to solve an equivalent boundary value problem in analytic function theory using available methods and then recapture the solution of the DP through an integration. Among the DP that are amenable to this approach are those whose associated problems unlock by singular integral equations [6, 9, 23], quadratures [9], Bergman kernels [1, 3], Riemann series expansions [18, 19], and the method of particular solutions [9, 11].

Regarding the method of particular solutions, one considers the error in approximating a solution u by an element in a finite dimensional subspace S of particular solutions. The natural question arises of finding the optimal function in S that approximates u with minimum error relative to a given norm. The related problem is to identify the best approximation map (if it exists) for the DP relative to the subspace S. The answers flow from non-linear convex programming problems. However, the algorithms for their solution are complicated and converge slowly as is the case for mini-max polynomial approximation of continuous functions on an interval or its analog in analytic function theory. Consequently, alternate methods are sought for finding information about the best map and/or the optimal solution.

The place to begin is with the theory of near-best approximation being developed by J. C. Mason [12–16] and others [4, 17] for analytic functions on a disk. Their basic idea is to replace the nonlinear problem of determining the optimal approximate of an analytic function relative to a given subspace and fixed norm with that of solving an appropriately defined linear problem. The method is constructive so that the results are useful when the errors in making this replacement are acceptably small. The aim here is to recast this theory into a function theoretic setting and thereby obtain practical information about approximate solutions of the DP. This imbedding is accomplished by modifying and inverting operators of S. Bergman [1] and R. P. Gilbert [7, 9] to construct an integral transform pair with norm preserving properties.

2. Preliminary Work

The radial form of Eqs. (1) that is traditionally taken in function theory on the disk is

$$[\nabla^2 + ra(r^2)\,\partial_r + c(r^2)]\,w = 0$$

with coefficients that are entire functions. It reduces to the Helmholtz equation when $a(r^2) \equiv 0$ and $c(r^2) = -\lambda^2 < 0$. Changing dependent variables by the relation

$$w := u \exp \left\{ (-\tfrac{1}{2}) \int_0^r a(t^2)\, dt \right\}$$

puts the equation into its canonical *radial form* [9],

$$\mathscr{L}(u) := [\nabla^2 + F(r^2)]\, u = 0, \tag{3}$$

where

$$F(r^2) = -r/2\, \partial_r a(r^2) - a(r^2) - r^2 a^2(r^2)/4 + c(r^2).$$

For convenience Eq. (3) will be referred to as the *diffusion equation* because of its connection with that problem. The coefficient $F(r^2)$ is taken as nonpositive so that the DP has a unique solution [9–11].

To extend analytic function theory from the disk, we consider regular (classical) solutions in $D_\rho: x^2 + y^2 < \rho^2$. Each solution is associated with a unique analytic function f in $D_{2\rho}$ [1, 5] by the transform

$$u(r, \theta) = Bf(z) := \int_{-1}^{+1} E(r^2, t)\, f(\sigma)\, d\mu(t),$$

where

$$z = re^{i\theta}, \qquad \sigma = z(1 - t^2)/2.$$

The measure of this transform is

$$d\mu(t) = (1 - t^2)^{-1/2}\, dt,$$

and its kernel is expressed as a Taylor's series

$$E(r^2, t) = 1 + \sum_{n=1}^{\infty} t^{2n} Q^{(2n)}(r^2) \tag{4}$$

that is analytic in $t \in [-1, +1]$ and entire in $r \in [0, \infty)$. The coefficients themselves are entire function solutions of the system of differential equations

$$\partial_{r^2}(Q^{(2)}(r^2)) + 2F(r^2) = 0, \qquad Q^{(0)}(r^2) = 1,$$

where for $n = 1, 2, 3, \ldots,$

$$(2n+1)\, \partial_{r^2}(Q^{(2n+2)}) + 2\partial_{r^2}(r^2 Q^{(2n)}) + F(r^2)\, Q^{(2n)} - n\partial_{r^2}(Q^{(2n)}) = 0, \tag{5}$$

and $n=0, 1, \ldots,$

$$Q^{(2n+2)}(r^2)\,|_{r=0}=0.$$

A complete set of particular solutions relative to uniform convergence on compacta of D_ρ [1, 3, 9] is given for all $n=0, 1, 2, \ldots$ by the functions

$$\Psi_n(r, \theta) := B(z^n)=e^{in\theta}J^{(n)}(r^2), \qquad \text{where } z=re^{i\theta},$$

and

$$J^{(n)}(r^2) := (r/2)^n\, G_n(r^2),$$

where

$$G_n(r^2) := \int_{-1}^{+1} E(r^2, t)(1-t^2)^n\, d\mu(t).$$

It is convenient in this analysis to *normalize* these functions as

$$\Phi_n(r, \theta) := e^{in\theta}J^{(n)}(r^2)/J^{(n)}(\rho^2), \qquad n=0, 1, 2 \ldots, \tag{6}$$

so that $\Phi_n(\rho, \theta)=e^{in\theta}$. Here and throughout the remainder of this paper it is tacitly assumed that the radius ρ of the disk is selected so that it is not a zero of any $J^{(n)}$. If it were, the subsequent analysis is extendable to subspaces derived from compability conditions defined at the boundary of the disk and is left to the reader. We note that the estimate in Eq. (11), to follow, shows that ρ cannot be a limit point of zeros of a subsequence of the $J^{(n)}$'s.

The desire to formulate a near-best approximation theory in a natural way suggests an alternate to the operators found in [1, 8, 9]. Proceeding in this direction we define the kernel

$$K(s, r, \rho) := \lim_{m\to\infty} K_m(s, r, \rho),$$

where

$$K_m(s, r, \rho) := (1/2\pi i) \sum_{k=0}^{m} s^{k-1}J^{(k)}(r^2)/J^{(k)}(\rho^2), \qquad m=0, 1, 2, \ldots.$$

The particular solutions are then formally generated for all $n=0, 1, 2, \ldots$ by the integral transform

$$\Phi_n(r, \theta)=B_*(z^n) := \int_{|s|=1} K(s, r, \rho)\, \tau^n\, ds,$$

with generating variable $\tau = 2/s\rho$. The normalized form of S. Bergman's operator is now defined as

$$u(r, \theta) = B_* f(z) := \int_{|s|=1} K(s, r, \rho) f(\tau)\, ds \tag{7}$$

so that the regular solution

$$u(r, \theta) = \sum_{n=0}^{\infty} a_n \Phi_n(r, \theta), \qquad \text{where} \quad z = re^{i\theta}, \tag{8}$$

and the unique analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n z_n \tag{9}$$

correspond on D_ρ.

To verify this association, we begin by establishing convergence of the kernel. Let us proceed to factor it as follows:

$$E(r^2, t) - 1 = t^2 E^*(r^2, t). \tag{10}$$

Here the entire function $|E^*(s^2, t)| \leqslant 2c$, for some constant $c > 0$ provided that $s \leqslant \rho$. And, the estimate

$$\left| \int_{-1}^{+1} t^2 E^*(s^2, t)(1 - t^2)^{n - 1/2}\, dt \right| \leqslant c\omega_n/(n+1)$$

follows where

$$\omega_n := [2\Gamma(1/2)\, \Gamma(n + 1/2)]/\Gamma(n+1).$$

Working with Eq. (10) gives the estimate

$$|(G_n(s^2)/\omega_n) - 1| \leqslant c/(n+1)$$

which is uniform in $s \leqslant \rho$ for some conveniently chosen constant c. Let sufficiently small $\varepsilon > 0$ be given. From the above bound we see that there is an integer $n_0 = n_0(\varepsilon, \rho)$ such that for $n \geqslant n_0$ the bound

$$(1 - \varepsilon) \leqslant |G_n(s^2)/\omega_n| \leqslant (1 + \varepsilon) \tag{11}$$

is valid for all $s \leqslant \rho$. In other words, the ratios

$$|J^{(n)}(r^2)/J^{(n)}(\rho^2)| \leqslant M\varepsilon(r/\rho)^n, \qquad \text{where } r \leqslant \rho,$$

are bounded by terms that decrease geometrically on D_ρ. The method of dominates implies that the kernel $K(w, r, v)$ is absolutely and uniformly convergent on compacta of the set $(|w|=1)\times(|v|=\rho)$. Thus, the unique representation $u=B_*(f)$ is valid on D_ρ. We will find that the linear map B_* may be thought of as an isomorphism between the linear spaces of analytic functions in D_ρ and of regular solutions of the diffusion equation in D_ρ.

We turn our attention to the behavior of $B_*(f)$ at the boundary. In analytic function theory one conveniently takes the standard domain as the *unit disk* $\Delta := D_1$. Following Mason, let $\mathscr{A}(\Delta)$ be the linear space of analytic functions in Δ that is continuous (in the sense of radial limits) on the closure, $\mathrm{cl}(\Delta)$, of Δ. This is to say that if $f\in\mathscr{A}(\Delta)$, then f is analytic on Δ and

$$f(e^{i\theta}) := \lim_{r\uparrow 1} f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$$

is a (uniformly) continuous function on $[0, 2\pi)$. It is necessary to examine the continuity of $B_*(f)$ at the boundary and thereby identify the linear space $\mathscr{A}^*(\Delta) := \{B_*(f): f\in\mathscr{A}(\Delta)\}$ with the linear space $\mathscr{A}(\Delta)$. Proceeding in this direction we apply some of the methods in [1; 2, pp. 430–433]. Write

$$\begin{aligned} u(r,\theta) &:= [u(r,\theta) - f(re^{i\theta})] + f(re^{i\theta}) \\ &= \sum_{n=0}^{\infty} \{a_n r^n e^{in\theta}(G^{(n)}(r^2) - G^{(n)}(1))/G^{(n)}(1)\} + f(re^{i\theta}). \end{aligned}$$

To verify that the limit of the series is zero at the boundary, we first apply the triangle inequality and Eq. (11) to show that

$$|(G^{(n)}(r^2) - G^{(n)}(1))/\omega_n| \leqslant c_1/(n+1).$$

The next step is to rewrite the bracketed term and apply this estimate as follows:

$$\begin{aligned} |(G^{(n)}(r^2) - G^{(n)}(1))/G^{(n)}(1)| &\leqslant |((G^{(n)}(r^2) - G^{(n)}(1))/\omega_n)/G^{(n)}(1)/\omega_n| \\ &\leqslant (c_1/(n+1))\,|\omega_n/G^{(n)}(1)|. \end{aligned}$$

It then follows from the lower bound in Eq. (11) that

$$|(G^{(n)}(r^2) - G^{(n)}(1))/G^{(n)}(1)| \leqslant K/(n+1) \tag{12}$$

for all positive integers $n \geqslant M$ where M and K are suitable constants. Working with the tail of the series gives

$$\left|\sum_{n=m}^{p} a_n r^n e^{in\theta}(G^{(n)}(r^2) - G^{(n)}(1))/G^{(n)}(1)\right| \leqslant K \sum_{n=m}^{p} |a_n|/(n+1).$$

Successive applications of the CBS and Bessel's inequalities show that

$$\sum_{n=m}^{p} |a_n|/(n+1) \leqslant (1/4\pi^2) \int_{\partial\Delta} |f(s)|^2\, ds \sum_{n=m}^{p} (1/(n+1)^2).$$

In other words, the series tends to zero independently of (r, θ) and therefore $u(r, \theta) \to f(e^{i\theta})$ uniformly for $\theta \in [0, 2\pi)$ as $r \to 1$. Because the identification of u with $B_*(f)$ is unique, the spaces $\mathscr{A}^*(\Delta)$ and $\mathscr{A}(\Delta)$ are isomorphic. This means that the functions in $\mathscr{A}^*(\Delta)$ and solutions of the DP are naturally identified by the operator B_*.

The identification $u = B_* f = f$ at the boundary provides sufficient information to recover the associate of u. The inversion is simply provided by the Cauchy integral

$$f(re^{i\theta}) = B_*^{-1} u(r, \theta) := \int_{\partial\Delta} C(s, re^{i\theta})\, u(s)\, ds, \tag{13}$$

where

$$u^-(e^{i\theta}) := \lim_{r \uparrow 1} u(r, \theta)$$

and

$$C(s, z) := (1/2\pi i)(s - z)^{-1},$$

with the observation that the boundary function is the restriction of the associate to the boundary of Δ. We summarize these ideas in the following theorem.

THEOREM 1. *A regular solution u of the DP is uniquely represented by $u = B_* f$, where f is the analytic continuation of the boundary function to Δ. Conversely, the associate f is uniquely recovered from a regular solution u of the DP by $f = B_*^{-1} u^-$, where u^- is the restriction of u to the $\partial\Delta$.*

Let us turn in the direction of some applications that are related to the useful approximation of functions on a disk.

3. Function Theory of Near-Best Approximations

With the basic transformations in place, a few definitions are now reviewed. Let Y be a linear space endowed with a norm, $\|\cdot\|$, and let X be a subspace of Y. A map $M: Y \to X$ is referred to as an *approximation map.* If the map sends each element $y \in Y$ onto its best or optimal approximation $My = y_b \in X$ in the sense that

$$\|y - y_b\| \leqslant \|y - h\|, \qquad \text{for all } h \in Y,$$

then it is referred to as a *minimal* or *best* approximation map. Needless to say, the problem of finding a best approximation map can quickly become nonlinear [4]. To compensate for this, the idea of near-best approximation is introduced [13, 17]. An element $y_* \in X$ is a *near-best* approximation of $y \in Y$ to within a relative distance v if the estimate

$$\|y - y_*\| \leqslant (1 + v)\, \|y - y_b\| \tag{14}$$

applies. The usefulness of the estimate depends on v being sufficiently small.

There is an important class of approximation maps called the *projections*; these are bounded, linear, and idempotent. Let P: $Y \to X$ be a projection and define the operator norm $\|\cdot\|$ as

$$\|P\| := \sup\{\|Py\| : \|y\| = 1, y \in Y\}.$$

A projection has the property that

$$\|y - Py\| \leqslant \|I - P\| \operatorname{dist}(y, Y),$$

where $\operatorname{dist}(y, Y) := \inf\{\|y - t\| : t \in Y\}$ so that the operator inequality $\|I - P\| \leqslant 1 + \|P\|$ shows that

$$\|y - Py\| \leqslant (1 + \|P\|)\, \|y - y_b\| \tag{15}$$

whenever a best approximation y_b to y exists. The projection provides a near-best approximation $Py = y_*$ of y and $v = \|P\|$ gives a bound on the relative distance from the best approximation.

The plan is to develop aspects of near-best approximation in the space $\mathscr{A}^*(\Delta)$ from antecedents that pertain to the space $\mathscr{A}(\Delta)$. Consider an $(n+1)$ dimensional subspace

$$X := \operatorname{span}\{\phi_0, \phi_1, \phi_2, ..., \phi_n\} \subset \mathscr{A}(\Delta)$$

and its image

$$X^* := \operatorname{span}\{B_*\phi_0, B_*\phi_1, ..., B_*\phi_n\} \subset \mathscr{A}^*(\Delta)$$

under the isomorphism B_*. A map M: $\mathscr{A}(\Delta) \to X$ induces the natural approximation map $M_* = B_* M B_*^{-1}$: $\mathscr{A}^*(\Delta) \to X^*$. Conversely, the approximation map $M = B_*^{-1} M B_*$: $\mathscr{A}(\Delta) \to X$ is induced by the natural map $M_*\colon \mathscr{A}^*(\Delta)\to X^*$. In particular, the projection $P\colon \mathscr{A}(\Delta)\to X$ induces the natural map

$$P_* = B_* P B_*^{-1}\colon \mathscr{A}^*(\Delta)\to X^* \tag{16}$$

which is easily shown to be a projection. Since the map B_* is an isomorphism, the functions

$$P_* u(r,\theta)=\sum_{k=0}^{n} a_k B_* \phi_k(r,\theta)$$

and

$$Pf(re^{i\theta})=\sum_{k=0}^{n} a_k \phi_k(r,\theta) \tag{17}$$

are uniquely identified by the corresponding projections of $u=B_* f$.

In this paper we are working with the ordinary *uniform* (Chebyshev) *norm*

$$\|h\| := \max\{|h(z)| : z\in \mathrm{cl}(\Delta)\}$$

on $h\in\mathscr{A}(\Delta)\cup\mathscr{A}^*(\Delta)$. The map B_* is an isometry relative to this norm. To verify this useful property, let $u=B_* f$, and show that $\|u\|=\|B_* f\|$. On the spaces under consideration maximum principles hold. This means that $\|u\|$ and $\|f\|$ are attained on the boundary where u and f are identical. It follows that $\|u\|$ and $\|f\|$ are identical. In particular, $\|B_* f\|=1$ whenever $\|f\|=1$ proving that the operator norm $\|B_*\|=1$. Replacing $f=B_*^{-1}u$ and $u=B_* f$ in this reasoning establishes that the operator norm $\|B_*^{-1}\|=1$ is valid too. Application of these facts to the standard operator bounds $\|P_*\|\leqslant\|B_*\|\ \|P\|\ \|B_*^{-1}\|$ and $\|P\|\leqslant\|B_*^{-1}\|\ \|P\|\ \|B_*\|$ show that the induced projection is norm preserving, $\|P_*\|=\|P\|$. This information is summarized in the following theorem.

THEOREM 2. *Let X be a finite dimensional subspace of $\mathscr{A}(\Delta)$ and let P be a projection that sends $\mathscr{A}(\Delta)$ onto X. Then the natural induced projection $P^*=B_* P B_*^{-1}$ sends $\mathscr{A}^*(\Delta)$ onto X^* and preserves the norm, $\|P^*\|=\|P\|$.*

The result is clearly useful. A practical near-best approximation in the associated space $\mathscr{A}(\Delta)$ automatically induces a practical near-best approximation in the space $\mathscr{A}^*(\Delta)$. The best or optimal approximations correspond when they exist. Finally, a member P^0 of a set $\sum$ of projections with common domain and range that has minimum norm is referred to as a *minimal* or *best* projection map. We turn our attention to some examples.

5. Near-Best Approximation in Various Subspaces

Applications of these ideas to the DP will draw upon existing projection methods in the function theory of $\mathscr{A}(\Delta)$. The examples involve Taylor's series and the link between Lagrange interpolation and the discrete Fourier projection. Let us begin with subspaces X_n of $\mathscr{A}(\Delta)$ that are algebraic polynomials of degree n. These are designated by

$$X_n = \text{span}\{1, z, z^2, ..., z^n\}, \qquad \text{for fixed } n = 0, 1, 2, ...$$

Consider the set of projections

$$\sum (X_n) := \{P\text{: projections } P\text{: } \mathscr{A}(\Delta) \to X_n\}$$

and the set of induced projections

$$\sum (X_n^*) := \{P_*\text{: } P_* \text{ is a natural projection induced by } P \in \sum\}.$$

It is known [13, 17] that the minimal projection map $P^0\colon \mathscr{A}(\Delta) \to X_n$ exists. In view of the imbedding Theorem 1, the induced projection $P_*^0 := B_* P^0 B_*^{-1}$ is a minimal projection map. The first example shows that the *induced Taylor projection*

$$T_*^{(n)} = B_* T^{(n)} B_*^{-1} \tag{18}$$

is a minimal projection drawn from the Taylor projection map $T^{(n)}$ sending $f \in \mathscr{A}(\Delta)$ onto $T^{(n)}f$, the nth partial sum of the Taylor series expansion $T^{(n)}f$ of f. In the first section we constructed a natural identification of the solutions of the DP with the functions in $\mathscr{A}^*(\Delta)$. This leads to the first example.

THEOREM 3. *On the disk Δ, let u be the solution of the DP with continuous boundary data f. Then each of the following is valid*:

(a) *The induced Taylor series projection $T_*^{(n)}$ is the minimal projection in the family of projections $\sum (X_n^*)$ from the solutions of the DP onto X_n^* for fixed $n = 0, 1, 2, ...$.*

(b) *The induced Taylor series projection has bounded norm; $\|T_*^{(n)}u\| \leqslant \tau_n$ where*

$$\tau_n := (1/\pi) \int_0^\pi |\sin(n+1)t| / \sin t \sim (4/\pi^2)\, O(\log n)$$

for large n.

(c) *$T_*^{(n)}u$ is a near-best approximation of u in X_n^* for fixed $n = 0, 1, 2, \ldots$.*

(d) *The representation of the induced projection is*

$$T_*^{(n)}u(r, \theta) = \int_{\partial\Delta} K_*^{(n)}(re^{i\theta}, s)\, f(s)\, ds$$

with kernel

$$K_*^{(n)}(z, s) := (1/2\pi i)\, B_*\{(z^{n+1} - s^{n+1})/s^{n+1}(z - s)\}$$

for fixed $n = 0, 1, 2, \ldots$.

Proof. The verification of these claims is immediate. First, let P_* be a projection in $\sum(X_n^*)$. Then $\|P_*\| = \|P\|$ and since the projection $T^{(n)}$ is minimal in $\sum(X_n)$ (see [13, 17]), $\|P\| \geqslant \|T^{(n)}\|$. However, $\|T_*^{(n)}\| = \|T^{(n)}\|$ which completes the first part because then the bound $\|P_*\| \geqslant \|T_*^{(n)}\|$ shows that the induced map is a minimal projection in $\sum(X_n^*)$. Part (b) follows from the identity $\|T_*^{(n)}\| = \|T^{(n)}\| \leqslant \tau_n$ where τ_n is defined in [13]. Continuing this line of reasoning we will find that two best approximations correspond. They are the approximation u_b of $u \in \mathscr{A}^*(\Delta)$ and the associated approximation f_b whose existence is guaranteed [13, 17]. The correspondence is a direct consequence of the uniqueness of the representations and the isometric property of the operators displayed by the estimates; $\|u - u_b\| = \|B_*(f - f_b)\| \leqslant \|B_*\|\ \|f - h\| = \|f - h\| = \|B_*^{-1}u - B_*^{-1}v\| \leqslant \|B_*^{-1}\|\ \ \|u - v\| = \|u - v\|$, $v \in \mathscr{A}^*(\Delta)$. Note that $u_b \in \mathscr{A}^*(\Delta)$ because the map is closed.

Having found the best approximation of u, one sees that the induced Taylor map is a useful near-best approximation from the bound

$$\|u - T_*^{(n)}u\| \leqslant (1 + \|T_*^{(n)}\|)\ \|u - u_b\|,$$

and the estimate $\|T_*^{(n)}\| \sim (4/\pi^2)\ O(\log n)$. The proof is completed by finding the map. Let

$$\begin{aligned} T_*^{(n)}u(r, \theta) &= B_* T^{(n)} B_*^{-1} u(r, \theta) \\ &= B_* T^{(n)} f(z) = B_*\left(\sum_{k=0}^{n} a_k z^k\right), \qquad \text{where} \quad z = re^{i\theta}, \end{aligned}$$

for fixed $n = 0, 1, 2, \ldots$, where

$$a_k := (1/2\pi i) \int_{\partial\Delta} s^{-k-1} f(s)\, ds, \qquad k = 0, 1, 2, \ldots, n.$$

Rearranging some terms gives the representation of the map as

$$T_*^{(n)}u(r, \theta) = B_* \left\{ \int_{\partial \Delta} (1/2\pi i) \sum_{k=0}^{n} (z/s)^k s^{-1} f(s)\, ds \right\}$$

$$= \int_{\partial \Delta} B_* \{ (1/2\pi i)((1-(z/s)^{n+1})/s(1-(z/s))) \} f(s)\, ds.$$

We remark that on compacta of Δ the uniform convergence of $T^{(n)}f \to f$ induces the uniform convergence of $T_*^{(n)}u \to u$ as $n \to \infty$.

The second application of the imbedding theorem focuses on interpolation of the boundary values on a set $\Omega_n := \{z_0, z_1, ..., z_n\}$ of preassigned points. A Lagrange interpolating projection map is defined by $L_n: \mathscr{A}(\Delta) \to Y_n$ where $Y_n :=$ {polynomials interpolating f at Ω_n} (see [13, 17]). If $g \in Y_n$, then $B_* g$ is a generalized polynomial solution of the diffusion equation that interpolates boundary values

$$B_* g(z_j) = f(z_j), \qquad \text{where } 0 \leqslant j \leqslant n.$$

Of the set of Lagrange interpolating polynomials, there is a special subspace Z_n that interpolates at the *Fourier points* $\Delta_n := \{1, \omega, \omega^2, ..., \omega^{n-1}\}$, where ω is a primitive nth root of unity. These are the *discrete Fourier projections* defined for fixed $n = 0, 1, 2, ...$, by

$$F^{(n)}f(z) := \sum_{k=0}^{n} f(\omega^k) f_k(z),$$

where the fundamental polynomials of pointwise interpolation [17, 20] are

$$f_k(z) := (z^{n+1} - 1)/(n+1)\, \omega^{kn}(z - \omega), \qquad \text{where } 0 \leqslant k \leqslant n.$$

Like the case of the Taylor's projection, a natural Fourier projection is induced by $F_*^{(n)} = B_* F^{(n)} B_*^{-1}$. This generalized discrete Fourier projection must interpolate the boundary values of the solution of the DP by

$$F_*^{(n)}u(1, \theta_j) = f(\omega^j), \qquad e^{i\theta_j} = \omega^j, \qquad \text{where} \quad 0 \leqslant j \leqslant n. \tag{19}$$

And, it too is norm preserving, $\|F_*^{(n)}\| = \|F^{(n)}\|$. Let $\sum (Y_n)$ be the family of Lagrange interpolating projections $L_n: \mathscr{A}(\Delta) \to Y_n$. The family of induced interpolating projections is $\sum (Y_n^*)$. The minimality of the map $F_*^{(n)}$ is not proved, but we see that $F_*^{(n)}u$ is a useful near-best approximation for u in the subspace $Z_n^* := \{F_n^* : F_n^* = B_* F_n B_*^{-1},\ L_n^* : \mathscr{A}^*(\Delta) \to Z_n^*\}$ by exhibiting a bound for the norm $\|F_*^{(n)}\|$. This and a few more facts are brought out in the next theorem.

THEOREM 4. *On the disk Δ, let u be the solution of the DP with continuous boundary data f. Then each of the following is valid*:

(a) *The induced Fourier series projection $F_*^{(n)}u$ is a solution of the diffusion equation that is a near-best approximation of u interpolating f at the Fourier points for fixed $n = 0, 1, 2, \ldots$.*

(b) *The norm $\|F_*^{(n)}\| = \gamma_n$ where*

$$\gamma_n := (1/(n+1)) \sum_{k=0}^{n} \csc((2k+1)\pi/(2n+2)) \sim O(2/\pi \log n)$$

for large n.

(c) *Therefore, $F_*^{(n)}u$ is a useful near-best approximation of u for fixed $n = 0, 1, 2, \ldots$.*

(d) *The representation of the induced projection is*

$$F_*^{(n)}u(r, \theta) := \int_{\partial\Delta} C_*^{(n)}(s, re^{i\theta}) f(s)\, ds$$

with the kernel

$$C_*^{(n)}(s, z) := B_* \left\{ \sum_{k=0}^{n} C(s, \omega^k) f_k(z) \right\}$$

for fixed $n = 0, 1, 2, \ldots$.

Proof. The projection $F^{(n)}u$ is constructed as a solution of the diffusion equation. And, $F_*^{(n)}u$ interpolates f at the Fourier points because the identity is valid $F_*^{(n)}u = F^{(n)}f$ at the boundary. Moreover, the projection is a near-best approximation map since

$$\|u - F_*^{(n)}u\| \leqslant (1 + \|F_*^{(n)}\|)\, \|u - v_b\|,$$

where $v_b = B_* f_b$ for f_b which is the best approximation of u in the subspace of interpolating polynomials at the boundary [17]. The norm equality $\|F_*^{(n)}\| = \|F^{(n)}\| \leqslant \gamma_n$ establishes (b) and the fact that $F_*^{(n)}u$ is a useful near-best approximation of u because $\gamma_n \sim O(2/\pi \log n)$ (see [13, 17]).

To find the realization of the map let us compute

$$\begin{aligned} F_*^{(n)}u(r, \theta) &= B_* F^{(n)} B_*^{-1} u(r, \theta) \\ &= B_* F^{(n)} f(z) = B_* \left\{ \sum_{k=0}^{n} f(\omega^k) f_k(z) \right\}, \qquad \text{where } z = re^{i\theta}. \end{aligned}$$

At the Fourier points, the identity

$$f(\omega^k) = \int_{\partial\Delta} C(s, \omega^k) f(s)\, ds, \qquad \text{where} \quad 0 \leqslant k \leqslant n,$$

holds and the resulting equation

$$F_*^{(n)} u(r, \theta) = \int_{\partial\Delta} B_*^{-1} \left\{ \left(\sum_{k=0}^{n} C(s, \omega^k) f_k(re^{i\theta}) \right) \right\} f(s)\, ds$$

for the representation of the induced projection completes the argument.

It is useful to point out that numerical methods are available for aiding in calculations. These are inherited from the analytic function theory by the imbedding process. In particular, the Taylor coefficients of the "polynomial" $T_*^{(n)} u$ can be computed by numerical contour integration on the $\partial\Delta$ via Cauchy's formula. By trapezoidal discretization of the integral, the computation of the m-point trapezoidal discretization approximations to the mth Taylor coefficient reduces to the discrete Fourier transform of m values at the m roots of unity (see [17]). Doubling the number of function evaluations at each stage, an effective fast Fourier transform computes the sequence $F^{(m-1)}f$, $m = 1, 2, 4, \ldots,$ and, consequently computes the coefficients of $F_*^{(m-1)} u$ when the approximate solutions arise from the Taylor algorithm for $T^{(n)}f$. For numerical examples refer to [17].

5. Concluding Remarks

In considering function theoretic generalizations of these ideas for the diffusion equation, several possibilities unfold. We shall comment on some of these. First, to investigate an $\mathscr{L}^p(\Delta)$ theory for $p > 1$, meaningful criteria for the kernel of the operator to be in $\mathscr{L}^{p/(p-1)}(\Delta)$ are not easy to identify from the system of differential–difference equations (5) used to generate the Taylor coefficients. The operators B and B_* do not extend to doubly connected domains without modifications that render them ineffective. In looking at higher dimensional problems with axial symmetry, the problem remains two dimensional (two independent variables) even though the interpretation is higher dimensional. Transform theory of Gilbert [8, 9] and A. Weinstein [22, 23] show that for the axially symmetric Laplacian the DP extends to spaces E^n of (fractional) dimension $n \geqslant 3$ ($n \neq$ integer). We find that summability methods imply the imbedding theorem for $n \geqslant 3$. However, at this point, the transition from $n = 2$ to $n > 2$ remains open.

Thus, to bridge this gap is to extend the results of Mason *et al.* [12–17] to E^n $(n \geqslant 2)$. This will be the subject of a later report.

In closing, it should be noted that the uniform convergence of the associated analytic functions on compacta of Δ induces uniform convergence of the series expansions of the solution. These may be analytically continued as solutions of the diffusion equation on $C/\mathrm{cl}(\Delta)$ by Gilbert's envelope method.

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